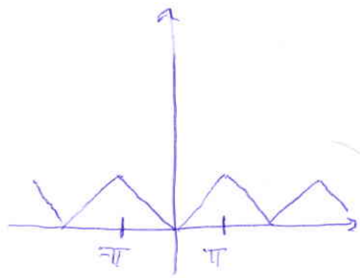
Exercice n°16 :

1) $f$ paire $\Rightarrow b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\overbrace{f(x)}^{\text{paire}}\overbrace{\sin(nx)}^{\text{impaire}}}_{\text{impaire}}\,dx = 0$




$a_0(f) = \frac{1}{\pi}\int_{-\pi}^{\pi}\overbrace{f(x)}^{\text{paire}}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$

$a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi}\overbrace{f(x)\cos(nx)}^{\text{paire}}\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$

$u(x) = x \Rightarrow u'(x) = 1$ ; $v(x) = \frac{1}{n}\sin(nx) \Leftarrow v'(x) = \cos(x)$.

$a_n(f) = \frac{2}{\pi}\left(\frac{1}{n}\left[x\sin(nx)\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right) = \frac{-2}{n\pi}\int_0^{\pi}\sin(nx)\,dx = -\frac{2}{n\pi}\left(\frac{-1}{n}\left[\cos(nx)\right]_0^{\pi}\right)$

$a_n(f) = \frac{2}{n^2\pi}\left((-1)^n - 1\right)$ ; $a_{2p}(f) = 0$

$a_{2p+1}(f) = \frac{-4}{(2p+1)^2\pi}$

La série de Fourier de $f$ est $\frac{a_0(f)}{2} + \sum\left(a_n(f)\cos(nx) + b_n(f)\sin(nx)\right)$.

c'est-à-dire : $\frac{\pi}{2} - \frac{4}{\pi} \times \sum \frac{1}{(2p+1)^2}\cdot\cos((2p+1)x)$

2) Via Dirichlet, la série de Fourier de $f$ CVS vers $f$ sur $\mathbb{R}$ (car $f$ est $C^1$ par morceaux et continue sur $\mathbb{R}$)

Ainsi $\forall x \in \mathbb{R}$, $\underset{|x|}{\underset{=}{f(x)}} = \frac{\pi}{2} - \frac{4}{\pi}\cdot\sum_{p=0}^{+\infty}\frac{1}{(2p+1)^2}\cos((2p+1)x)$

En particulier pour $x = 0$ :

$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{p=0}^{+\infty}\frac{1}{(2p+1)^2} \Rightarrow \boxed{\sum_{p=0}^{+\infty}\frac{1}{(2p+1)^2} = \frac{\pi^2}{8}}$

$\sum_{n=1}^{+\infty}\frac{1}{n^2} = \sum_{p=1}^{+\infty}\frac{1}{(2p)^2} + \sum_{p=0}^{+\infty}\frac{1}{(2p+1)^2} = \frac{1}{4}\sum_{n=1}^{+\infty}\frac{1}{n^2} + \frac{\pi^2}{8}$

$\Rightarrow \left(1 - \frac{1}{4}\right)\sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{\pi^2}{8}$ donc $\sum_{n=1}^{+\infty}\frac{1}{n^2} = \frac{4}{3}\cdot\frac{\pi^2}{8} = \frac{\pi^2}{6}$

3) Parseval $\Rightarrow \frac{a_0^2(f)}{4} + \frac{1}{2}\sum_{n=1}^{+\infty}\left(a_n^2(f) + b_n^2(f)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\overbrace{f^2(x)}^{\text{paire}}\,dx$

$= \frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{p=0}^{+\infty}\frac{1}{(2p+1)^4} = \frac{1}{\pi}\int_0^{\pi} x^4\,dx = \frac{\pi^2}{3}$

$\sum_{p=0}^{+\infty}\frac{1}{(2p+1)^4} = \frac{\pi^2}{8}\left(\frac{\pi^2}{3} - \frac{\pi^2}{4}\right) = \frac{\pi^4}{96}$

$\sum_{n=1}^{+\infty}\frac{1}{n^4} = \sum_{p=1}^{+\infty}\frac{1}{(2p)^4} + \sum_{p=0}^{+\infty}\frac{1}{(2p+1)^4} = \frac{1}{16}\sum_{p=1}^{+\infty}\frac{1}{n^4} + \frac{\pi^4}{96} = \left(1 - \frac{1}{16}\right)\sum_{n=1}^{+\infty}\frac{1}{n^4} = \frac{\pi^4}{96}$

$= \sum_{n=1}^{+\infty}\frac{1}{n^4} = \frac{16}{15}\cdot\frac{\pi^4}{96} = \frac{\pi^4}{90}$

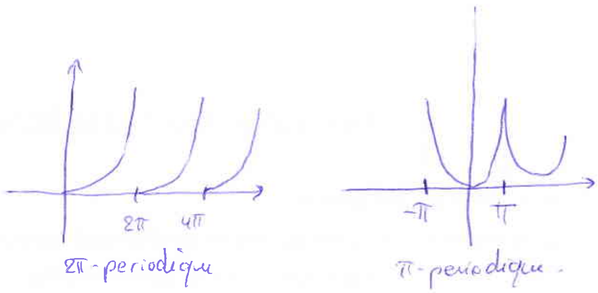
Exercice n°17 :



1) $f$ paire $\Rightarrow b_n(f)=0$

$$a_0(f)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \overset{\text{paire}}{x^2}\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2\pi^2}{3}.$$




$$a_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2\cos(nx)}_{\text{paire}}\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx$$

$u(x)=x^2 \Rightarrow u'(x)=2x$ ; $v(x)=\frac{1}{n}\sin(nx) \Leftarrow v'(x)=\cos(nx)$

$$a_n(f)=\frac{2}{\pi}\left(\frac{1}{n}\left[x^2\sin(nx)\right]_0^{\pi}-\frac{2}{n}\int_0^{\pi} x\sin(nx)\,dx\right)=\frac{-4}{n\pi}\int_0^{\pi} x\sin(nx)\,dx$$

$u(x)=x \Rightarrow u'(x)=1$ ; $v(x)=\frac{-1}{n}\cos(nx) \Leftarrow v'(x)=\sin(nx)$.

$$a_n(f)=\frac{-4}{n\pi}\left(\frac{-1}{n}\left[x\cos(nx)\right]_0^{\pi}+\frac{1}{n}\underbrace{\int_0^{\pi}\cos(nx)\,dx}_{=0}\right)=\frac{4}{n^2\pi}\cdot\pi(-1)^n=\frac{4(-1)^n}{n^2}$$

La série de Fourier de $f$ est donc $\underbrace{\frac{\pi^2}{3}}_{\frac{a_0(f)}{2}}+4\sum\frac{(-1)^n}{n^2}\cos(nx)$

2) Via dirichlet, comme $f$ est $C^1$ par morceaux et continue sur $\mathbb{R}$, la série de Fourier de $f$ CVS vers $f$ sur $\mathbb{R}$. Donc $\forall x\in\mathbb{R}$.

$$x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

En particulier pour $x=0$ : $0=\frac{\pi^2}{3}+4\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}$

donc $\boxed{\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2}=\frac{-\pi^2}{12}}$

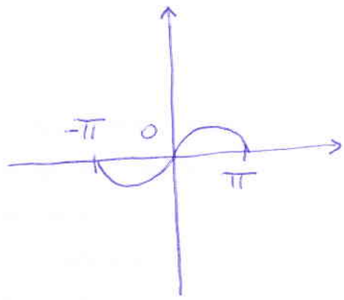
3) Parseval donne :

$$\frac{a_0^2(f)}{4}+\frac{1}{2}\sum_{n=1}^{+\infty}\left(a_n^2(f)+b_n^2(f)\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\overset{\text{paire}}{x^4}\,dx \Leftrightarrow \frac{\pi^4}{9}+8\sum_{n=1}^{+\infty}\frac{1}{n^4}=\frac{1}{\pi}\int_0^{\pi}x^4\,dx=\frac{\pi^4}{5}$$

$$\Rightarrow \sum_{n=1}^{+\infty}\frac{1}{n^4}=\frac{1}{8}\left(\frac{\pi^4}{5}-\frac{\pi^4}{9}\right)=\frac{\pi^4}{90}$$

## Exercice n°13 :

$$b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{\underbrace{f(x)}_{\text{impaire}}\underbrace{\sin(nx)}_{\text{impaire}}}^{\text{paire}}\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$



$$b_n(f) = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)\,dx \Leftarrow \boxed{\text{polynome de degré 2} = \text{double IPP}}$$

$u(x) = x(\pi - x) \Rightarrow u'(x) = \pi - 2x$

$v(x) = -\frac{1}{n}\cos(nx) \Leftarrow v'(x) = \sin(xn)$

$$b_n(f) = \frac{2}{\pi}\left(-\frac{1}{n}\underbrace{\left[x(\pi - x)\cos(nx)\right]_0^{\pi}}_{=0} + \frac{1}{n}\int_0^{\pi}(\pi - 2x)\cos(nx)\,dx\right)$$

$f$ impaire $\Rightarrow a_n(f) = 0$.

$$b_n(f) = \frac{2}{n\pi}\int_0^{\pi}(\pi - 2x)\cos(nx)\,dx\,.$$

$u(x) = \pi - 2x \quad u'(x) = -2$

$v(x) = \frac{1}{n}\sin(nx) \quad v'(x) = \cos(nx)$.

$$b_n(f) = \frac{2}{n\pi}\left(\frac{1}{n}\left[(\pi - 2x)\sin(xn)\right]_0^{\pi} + \frac{2}{n}\int_0^{\pi}\sin(nx)\,dx\right)$$

$$= \frac{4}{n^2\pi}\left(-\frac{1}{n}\left[\cos(nx)\right]_0^{\pi}\right) = \frac{4}{n^3\pi}\left((-1)^n - 1\right)$$

$b_{2p}(f) = 0$ ; $b_{2p+1}(f) = \frac{8}{(2p+1)^3\pi}$. La série de Fourier de $f$ est : $\frac{8}{\pi}\sum \frac{1}{(2p+1)^3}\sin((2p+1)x)$

$$\left(= \frac{-\cancel{8}4}{\pi}\sum \frac{1}{n^3}\left((-1)^n - 1\right)\sin(nx)\right)$$

2) Via Dirichlet, comme $f$ est $C^1$ par morceaux et continue sur $\mathbb{R}$ (car $C^1$ !), la série de Fourier de $f$ CVS vers $f$ sur $\mathbb{R}$.

En particulier $\forall x \in [0, \pi]$, $\underbrace{f(x)}_{=\,(x(\pi - x))} = \frac{8}{\pi}\sum_{p=0}^{+\infty}\frac{1}{(2p+1)^3}\sin((2p+1)x)$

En particulier pour $x = \frac{\pi}{2}$, on a :

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\sum_{p=0}^{+\infty}\frac{(-1)^p}{(2p+1)^3} \Longleftrightarrow \boxed{\sum_{p=0}^{+\infty}\frac{(-1)^p}{(2p+1)^3} = \frac{\pi^3}{32}}$$

Via Parseval (1755 - 1836),

$$\frac{a_0^2(f)}{4} + \frac{1}{2}\sum_{n=1}^{+\infty}\left(a_n^2(f) + b_n^2(f)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{f^2(x)}_{\text{paire}}$$

$\frac{32}{\pi^2}$

donc : $\frac{1}{2}\cdot\frac{64}{\pi^2}\sum_{p=0}^{+\infty}\frac{1}{(2p+1)^6} = \frac{1}{\pi}\int_0^{\pi}x^2(\pi - x)^2\,dx$ or $\int_0^{\pi}x^2(\pi - x)^2\,dx = \int_0^{\pi}x^2(\pi^2 - 2\pi x + x^2)\,dx$

$$= \pi^2\cdot\frac{\pi^3}{3} - 2\pi\cdot\frac{\pi^4}{4} + \frac{\pi^5}{5} = \pi^5\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) = \frac{\pi^5}{30}$$

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Donc

$$\frac{32}{\pi^2} \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^6} = \frac{\pi^4}{30} \Rightarrow \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^6} = \frac{\pi^6}{960}.$$

$$\sum_{p=0}^{+\infty} \frac{1}{n^6} = \sum_{p=1}^{+\infty} \frac{1}{(2p)^6} + \sum_{p=0}^{+\infty} \frac{1}{(2p+1)^6} = \frac{1}{64} \sum_{n=1}^{+\infty} \frac{1}{n^6} + \frac{\pi^6}{960}$$

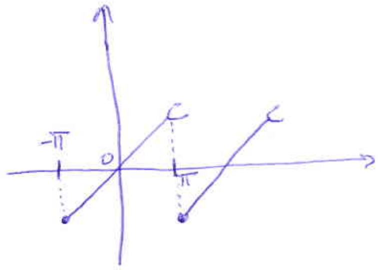
$$\underbrace{\left(1 - \frac{1}{64}\right)}^{\frac{63}{64}} \sum_{n=1}^{+\infty} \frac{1}{n^6} = \frac{\pi^6}{960} \Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^6} = \frac{64}{63} \cdot \frac{\pi^6}{960}$$

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$$

$$= \langle e_n, f \rangle$$

où $e_n : \mathbb{R} \to \mathbb{C}$, $x \mapsto e^{inx}$

Exercice n° 19.




$$a_0(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = 0.$$

$$a_n(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = 0.$$

$$b_n(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\,dx$$

$u(x) = x \Rightarrow u'(x) = 1$ ; $v(x) = \frac{-1}{n}\cos(nx) \Leftarrow v'(x) = \sin(nx)$

$$b_n(f) = \frac{2}{\pi}\left(\frac{-1}{n}\left[x\cos(nx)\right]_0^{\pi} + \frac{1}{n} \underbrace{\int_0^{\pi} \cos(nx)\,dx}_{=0}\right) = \frac{-2}{n\pi} \cdot \pi \cdot (-1)^n = \frac{2(-1)^{n+1}}{n}$$

La série de Fourier de $f$ est : $\frac{a_0(f)}{2} + \sum \left(a_n(f)\cos(nx) + b_n(f)\sin(nx)\right)$

$$= 2 \sum \frac{(-1)^{n+1}}{n} \sin(nx)$$

2) $\sin\left(n\frac{\pi}{2}\right) = \begin{cases} \text{si } n = 2k \Rightarrow 0 \\ \text{si } n = 2k+1 \Rightarrow (-1)^k \end{cases}$

3) $f$ est $C^1$ par morceaux et $C^0$ sur $]-\pi ; \pi[$. Donc $\forall x \in ]-\pi, \pi[$, $\underset{=x}{f(x)} = 2 \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$

En particulier pour $x = \frac{\pi}{2}$.

$$\frac{\pi}{2} = 2 \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} \sin\left(n\frac{\pi}{2}\right) = \frac{2}{1} \sum_{k=0}^{+\infty} \frac{(-1)^{2k+1}}{2k+1} \times \underbrace{(-1)^k}_{= \sin\left((2k+1)\frac{\pi}{2}\right)}$$

$$= 2 \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} \Rightarrow \boxed{\sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}}$$

4) Via Parseval

$$\frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{+\infty} \left(a_n^2(f) + b_n^2(f)\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^2(x)\,dx = \frac{1}{\pi} \int_0^{\pi} f^2(x)\,dx = \frac{1}{\pi} \int_0^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

donc $\frac{1}{2} \cdot 4 \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{3} \Rightarrow \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$