

EPITA

Mathematics

Midterm exam (S4)

March 2018

Name :

First name :

Class :

MARK :

Midterm exam

Duration : three hours
Documents and calculators not allowed

Exercise 1 (4 points)

Determine the nature of the following improper integrals :

1. $\int_0^1 \frac{\sqrt{1+t}-1}{t^3} dt.$

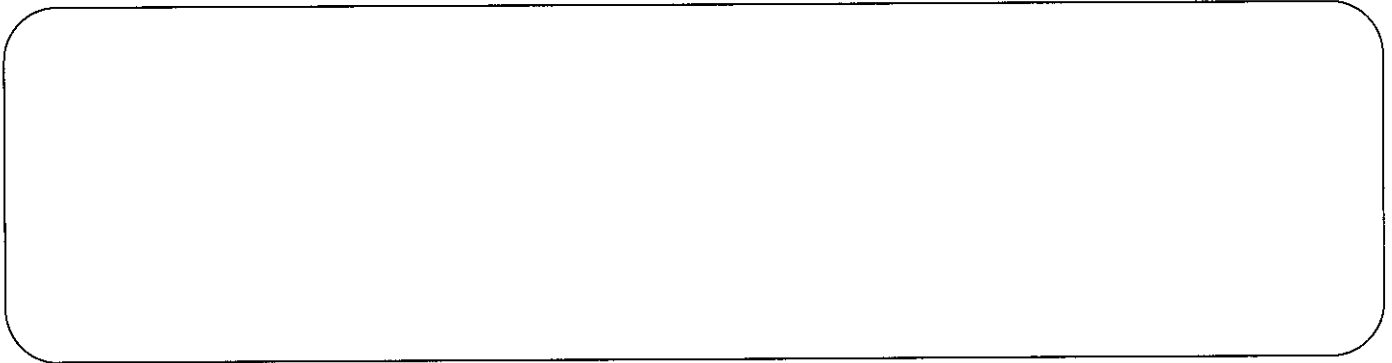
2. $\int_0^{+\infty} \frac{e^{-t}}{1+t^2} dt.$

3. $\int_0^{+\infty} \frac{1}{(1+t^2)\sqrt{t}} dt.$

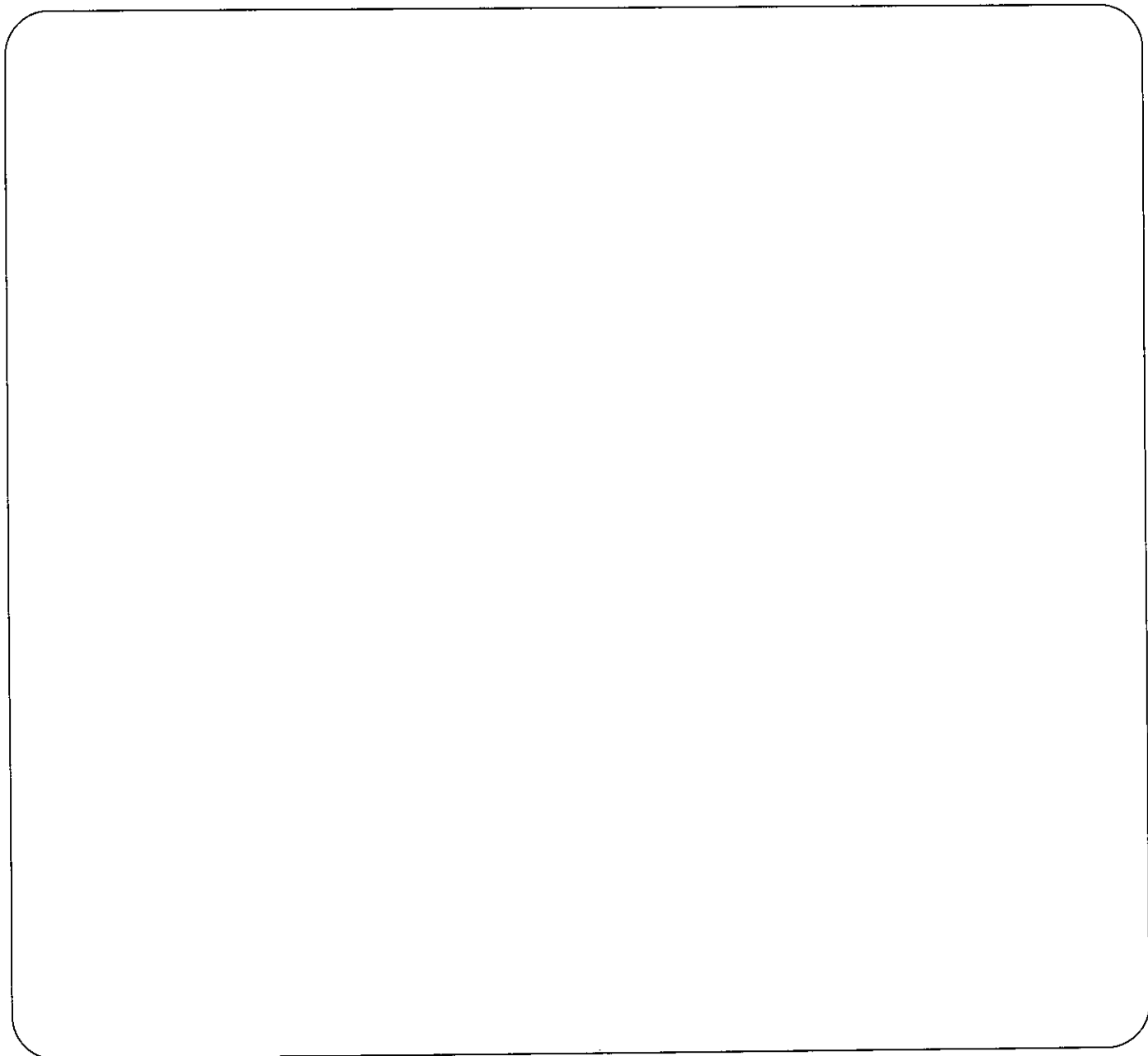
Exercise 2 (3 points)

Let $I = \int_0^{+\infty} \frac{dt}{(1+t^2)(1+t^n)}$ with $n \in \mathbb{N}$.

1. Prove that I is convergent.



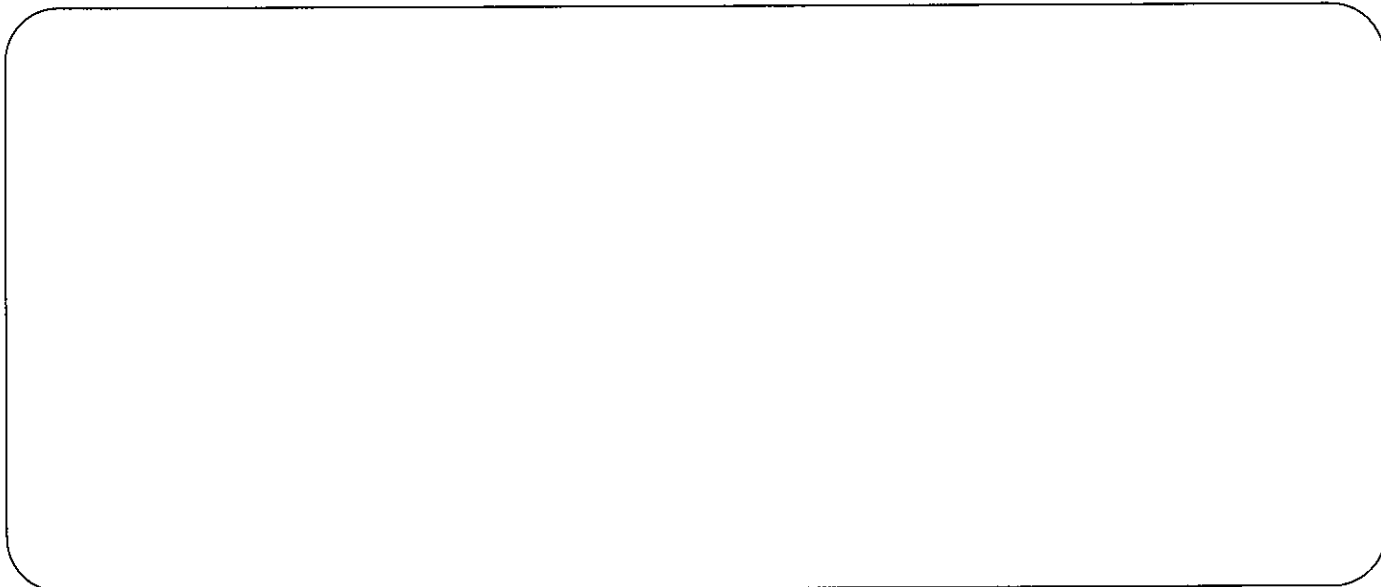
2. Using the substitution $u = \frac{1}{t}$ and using (after the substitution) that $u^n = 1 + u^n - 1$, calculate I .



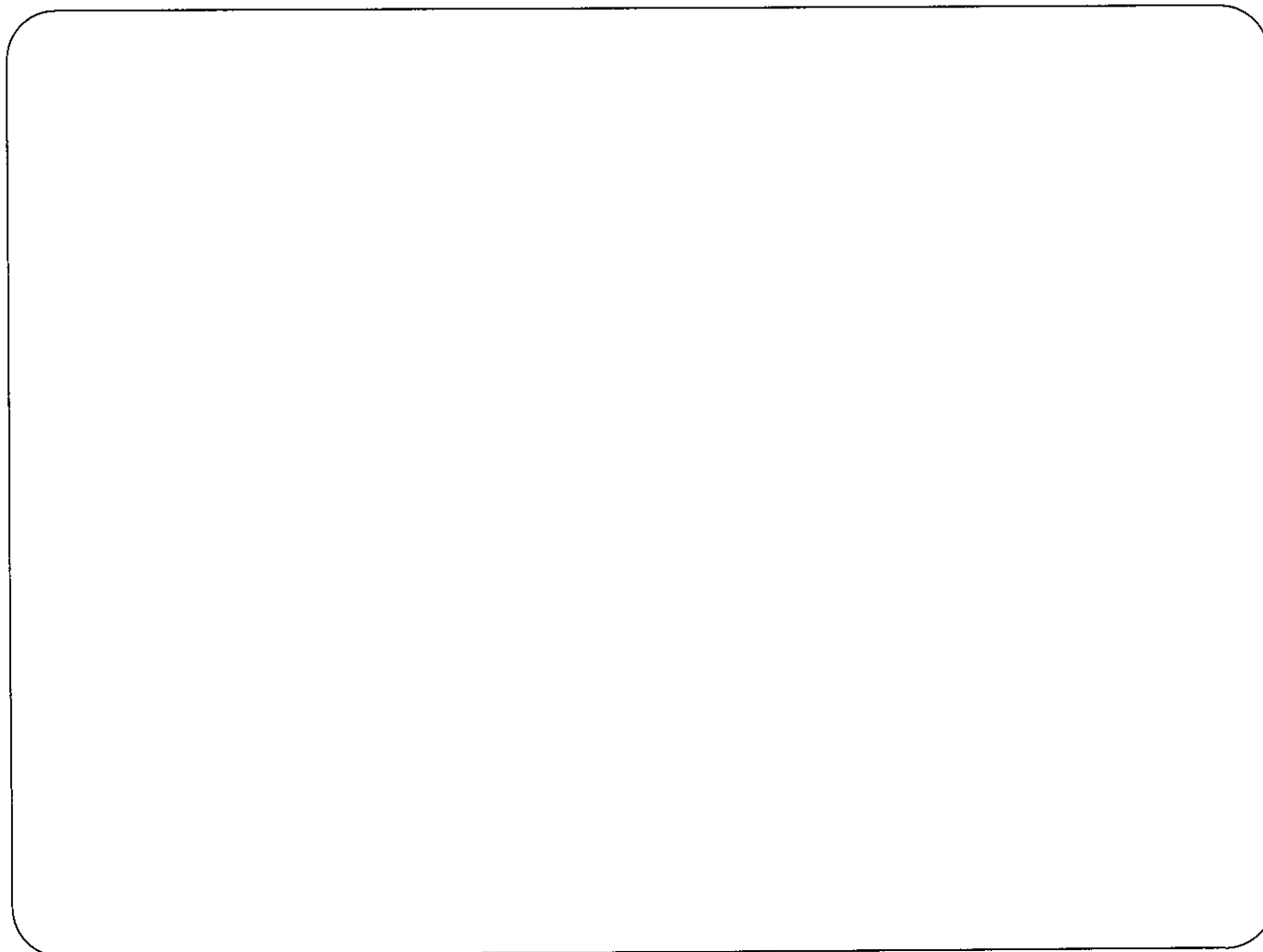
Exercise 3 (4 points)

Let $I = \int_0^{\frac{\pi}{2}} \ln(\sin(x)) dx$ and $J = \int_0^{\frac{\pi}{2}} \ln(\cos(x)) dx$.

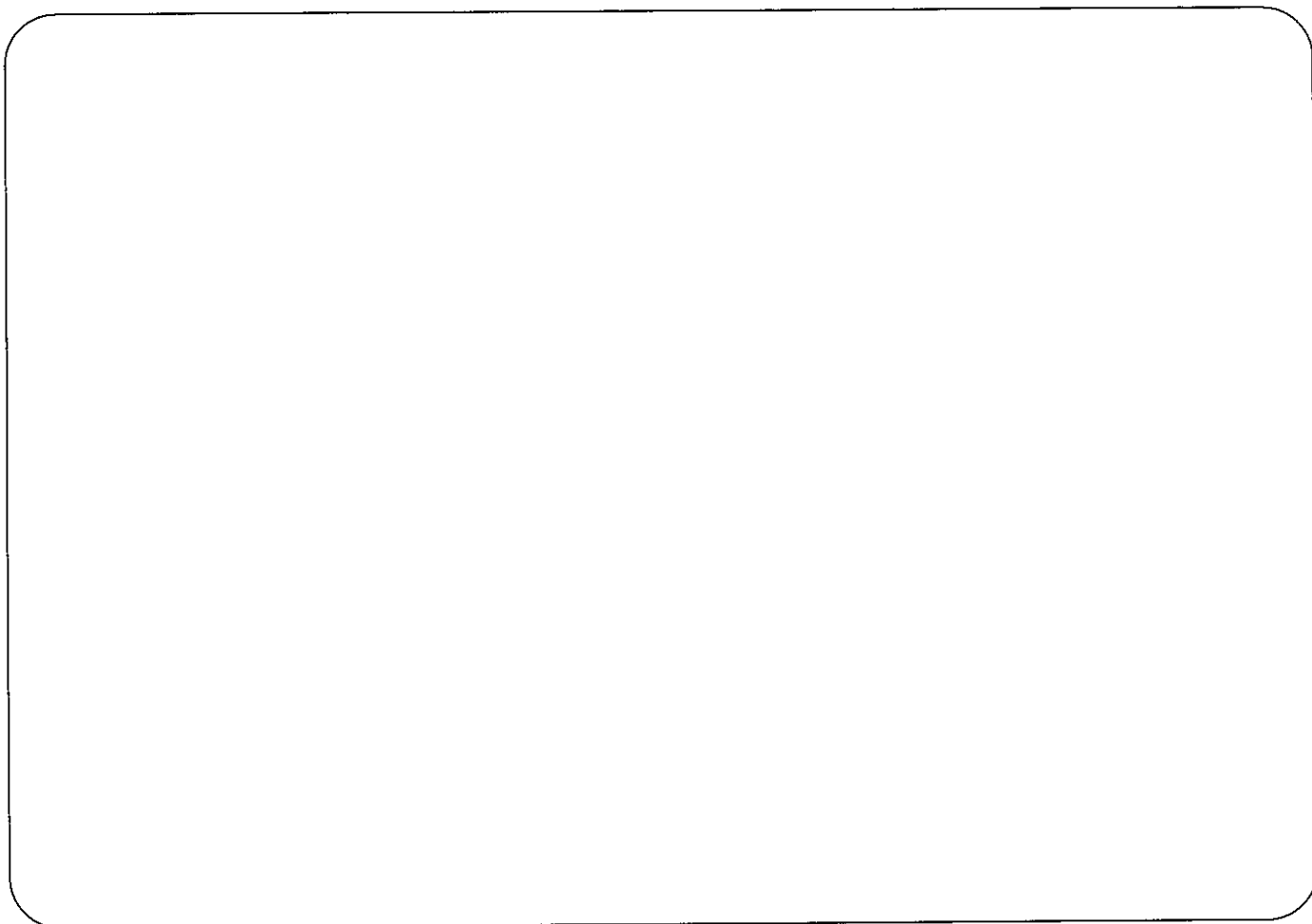
1. Show (with accuracy) that $\ln(\sin(x)) \underset{0}{\sim} \ln(x)$.



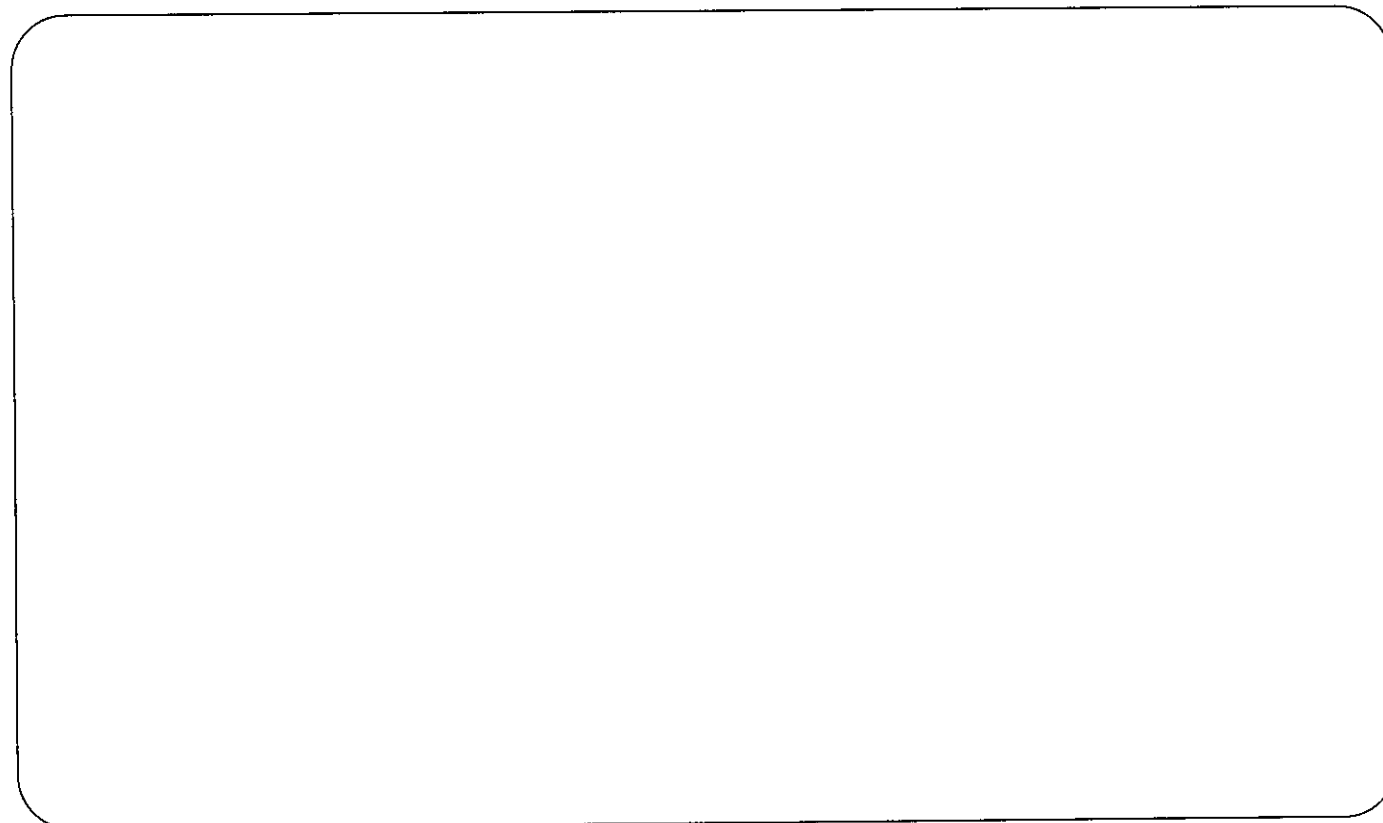
2. Show that I is convergent and, using the substitution $u = \frac{\pi}{2} - x$, that $I = J$.



3. Using the substitution $u = 2x$, show that $I = \int_0^{\frac{\pi}{2}} \ln(\sin(2x)) dx$.



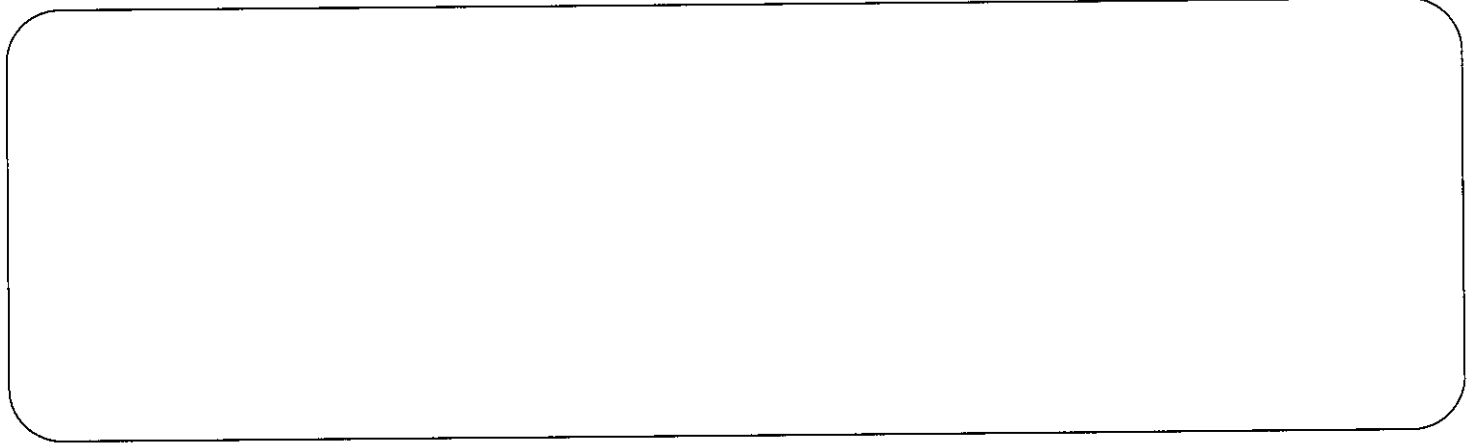
4. Using the relation $\sin(2x) = 2 \sin(x) \cos(x)$, deduce the value of I .



Exercise 4 (3,5 points)

Let $E = \mathbb{R}_2[X]$ together with the inner product $\langle P, Q \rangle = \int_{-1}^1 P(x)Q(x)(1-x^2) dx$. Using the Gram-Schmidt process starting with the basis $(1, X, X^2)$ of E , determine an orthogonal basis (P_0, P_1, P_2) of E (with respect to \langle, \rangle).

[this frame continues on the next page]

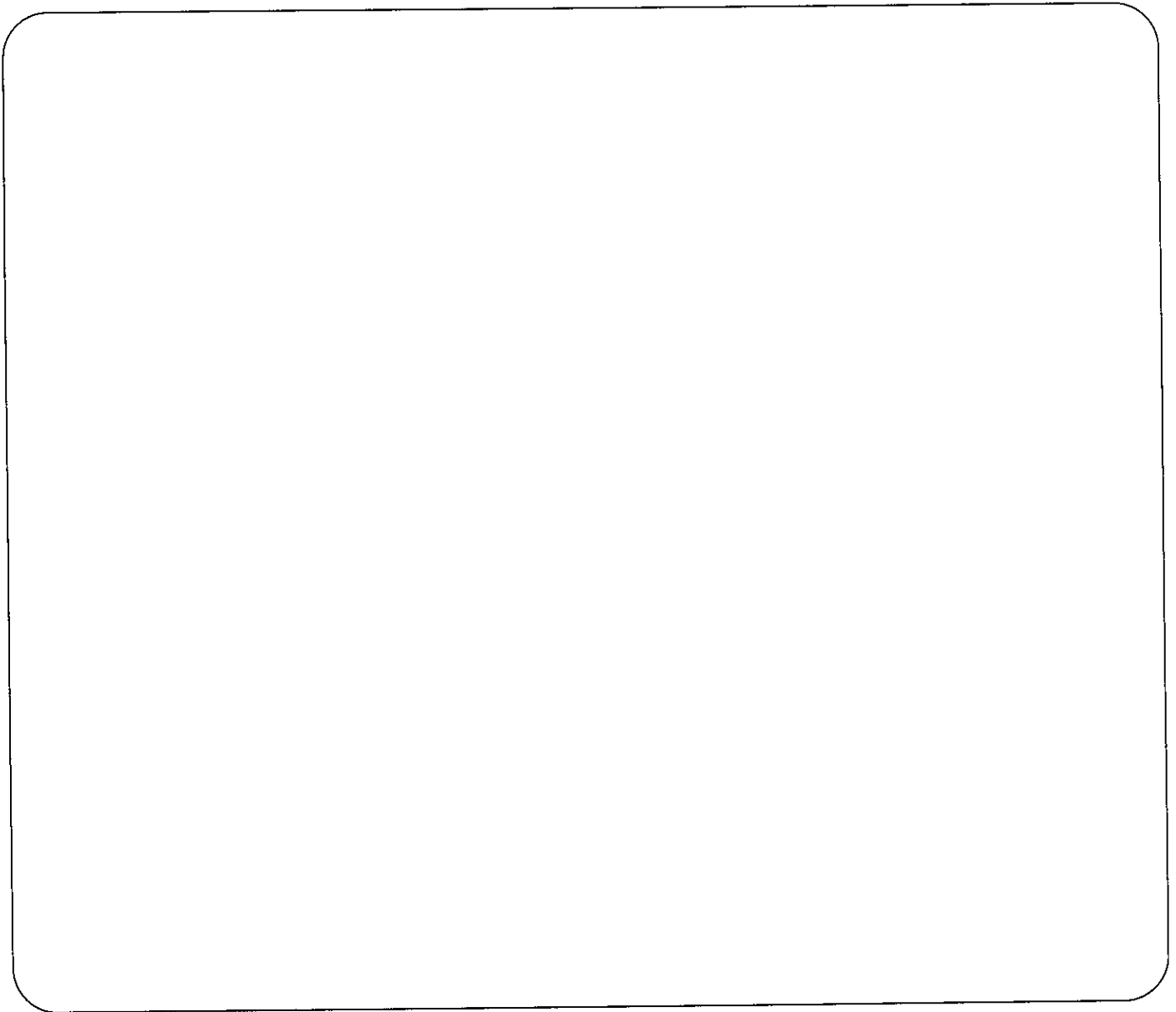


Exercise 5 (3 points)

Let (E, \langle, \rangle) be a Euclidean space and let $f : E \rightarrow E$ be an application.

1. Let us suppose that f checks the following property : $\forall (x, y) \in E^2 : \langle f(x), y \rangle = -\langle x, f(y) \rangle$. Show that

$$\forall (x, y, z) \in E^3, \forall \lambda \in \mathbb{R} : \langle f(\lambda x + y) - (\lambda f(x) + f(y)), z \rangle = 0$$



2. Show that the two following assertions are equivalent :

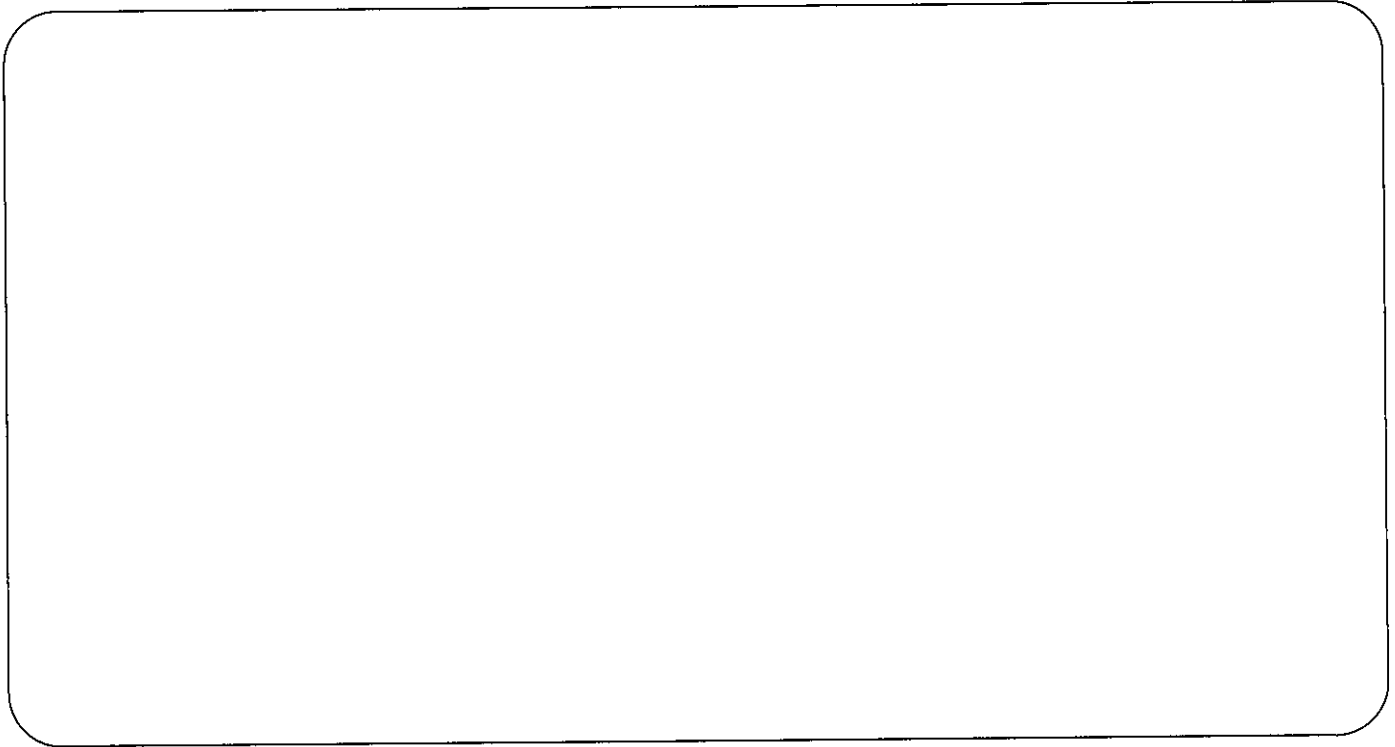
$$(i) \quad \forall (x, y) \in E^2 : \langle f(x), y \rangle = -\langle x, f(y) \rangle$$

$$(ii) \quad f \in \mathcal{L}(E) \text{ and } \forall x \in E \quad \langle f(x), x \rangle = 0$$

Exercise 6 (3 points)

Let $I = \int_0^{+\infty} \frac{t^2 + 1}{t^4 + 1} dt$.

1. Using the substitution $u = \frac{t}{\sqrt{2}}$, determine $\int_0^{+\infty} \frac{dt}{t^2 + 2}$. Deduce the value of $\int_{-\infty}^{+\infty} \frac{dt}{t^2 + 2}$.



2. Using the substitution $u = t - \frac{1}{t}$, calculate I .

