EPITA

Mathematics

Midterm exam (S4)

March 2018

Name :

First name :

Class :

MARK :

Midterm exam

Duration : three hours
Documents and calculators not allowed

Exercise 1 (4 points)

Determine the nature of the following improper integrals :

1. $\displaystyle\int_0^1 \frac{\sqrt{1+t}-1}{t^3}\,\mathrm{d}t.$

2. $\displaystyle\int_0^{+\infty} \frac{e^{-t}}{1+t^2}\,\mathrm{d}t.$

3. $\displaystyle\int_0^{+\infty} \frac{1}{(1+t^2)\sqrt{t}}\,\mathrm{d}t.$

Exercise 2 (3 points)

Let $I = \displaystyle\int_0^{+\infty} \frac{\mathrm{d}t}{(1+t^2)(1+t^n)}$ with $n \in \mathbb{N}$.

1. Prove that I is convergent.

2. Using the substitution $u = \dfrac{1}{t}$ and using (after the substitution) that $u^n = 1 + u^n - 1$, calculate I.

Exercise 3 (4 points)

Let $I = \int_0^{\frac{\pi}{2}} \ln\big(\sin(x)\big)\mathrm{d}x$ and $J = \int_0^{\frac{\pi}{2}} \ln\big(\cos(x)\big)\mathrm{d}x$.

1. Show (with accuracy) that $\ln\big(\sin(x)\big) \underset{0}{\sim} \ln(x)$.

2. Show that I is convergent and, using the substitution $u = \frac{\pi}{2} - x$, that $I = J$.

3. Using the substitution $u = 2x$, show that $I = \displaystyle\int_0^{\frac{\pi}{2}} \ln\bigl(\sin(2x)\bigr)\,\mathrm{d}x$.

4. Using the relation $\sin(2x) = 2\sin(x)\cos(x)$, deduce the value of I.

Exercise 4 (3,5 points)

Let $E = \mathbb{R}_2[X]$ together with the inner product $\langle P , Q\rangle = \displaystyle\int_{-1}^{1} P(x)Q(x)\big(1-x^2\big)\,\mathrm{d}x$. Using the Gram-Schmidt process starting with the basis $(1, X, X^2)$ of E, determine an orthogonal basis (P_0, P_1, P_2) of E (with respect to $\langle\, ,\rangle$).

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Exercise 5 (3 points)

Let $(E, \langle , \rangle)$ be a Euclidean space and let $f : E \to E$ be an application.

1. Let us suppose that f checks the following property : $\forall (x, y) \in E^2 \; : \; \langle f(x) \, , y \rangle = -\langle x \, , f(y) \rangle$. Show that

$$\forall (x, y, z) \in E^3, \forall \lambda \in \mathbb{R} \; : \; \Big\langle f(\lambda x + y) - \big(\lambda f(x) + f(y)\big) \, , z \Big\rangle = 0$$

2. Show that the two following assertions are equivalent :

$$(i)\ \ \forall (x,y) \in E^2 :\ \langle f(x)\,,y\rangle = -\langle x\,,f(y)\rangle$$

$$(ii)\ \ f \in \mathscr{L}(E) \text{ and } \forall x \in E\ \ \langle f(x)\,,x\rangle = 0$$

Exercise 6 (3 points)

Let $I = \displaystyle\int_0^{+\infty} \frac{t^2+1}{t^4+1}\,\mathrm{d}t$.

1. Using the substitution $u = \dfrac{t}{\sqrt{2}}$, determine $\displaystyle\int_0^{+\infty} \frac{\mathrm{d}t}{t^2+2}$. Deduce the value of $\displaystyle\int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{t^2+2}$.

2. Using the substitution $u = t - \dfrac{1}{t}$, calculate I.